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RESEARCH ARTICLE

REVISED Superradiant lasing in inhomogeneously broadened ensembles with spatially varying coupling

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Abstract

Background: Theoretical studies of superradiant lasing on optical clock transitions predict a superb frequency accuracy and precision closely tied to the bare atomic linewidth. Such a superradiant laser is also robust against cavity fluctuations when the spectral width of the lasing mode is much larger than that of the atomic medium. Recent predictions suggest that this unique feature persists even for a hot and thus strongly broadened ensemble, provided the effective atom number is large enough.
Methods: Here we use a second-order cumulant expansion approach to study the power, linewidth and lineshifts of such a superradiant laser as a function of the inhomogeneous width of the ensemble including variations of the spatial atom-field coupling within the resonator.
Results: We present conditions on the atom numbers, the pump and coupling strengths required to reach the buildup of collective atomic coherence as well as scaling and limitations for the achievable laser linewidth.
Conclusions: We show how sufficiently large numbers of atoms subject to strong optical pumping can induce synchronization of the atomic dipoles over a large bandwidth. This generates collective stimulated emission of light into the cavity mode leading to narrow-band laser emission at the average of the atomic frequency distribution. The linewidth is orders of magnitudes smaller than that of the cavity as well as the inhomogeneous gain broadening and exhibits reduced sensitivity to cavity frequency noise.

Keywords

superradiant laser, superradiance, bad-cavity laser, cumulant expansion, inhomogeneous broadening, cavity dephasing

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REVISED **Amendments from Version 1**

In the revised version, we have addressed the reviewers' comments on the original manuscript. In particular, Section 2 has been complemented by Figure 3 presenting the influence of noise on the laser linewidth. The beginning of Section 3 has been expanded with a brief discussion of factors that may cause inhomogeneous broadening of atomic frequencies. In addition to the main results, we have included an appendix regarding the cross-correlations between atoms in different clusters.

Any further responses from the reviewers can be found at the end of the article

Introduction

Collective stimulated emission of coherent light by atoms inside an optical cavity is a fundamental phenomenon studied for decades in quantum optics[1–7]. Even very recently a large number of theoretical and experimental studies focused on continuous superradiance[6–15], aiming at the development of a superradiant laser[16–22]. Such a superradiant laser typically operates in a bad-cavity regime, where the cavity mode is much broader than the natural linewidth of the atoms providing the gain. In the limit of low photon number operation the coherence necessary for frequency stability is stored in the atoms rather than the cavity field. This makes the laser frequency insensitive to thermal and mechanical fluctuations of the cavity, which is the main limitation for conventional good-cavity lasers[23,24]. In recent years pulsed superradiance has been experimentally demonstrated[16,20–22] and a number of new theoretical ideas have been proposed[25–27]. However, the experimental realization of a continuous wave superradiant laser has not yet been achieved.

Effects such as frequency broadening in the gain medium are an inherent part of any experiment. Such processes are capable of disrupting the collective interaction between the atoms and the cavity field. In this work, we aim to offer a comprehensive study of these potentially detrimental effects. To this end, we study a model of a superradiant laser and focus on inhomogeneity among the atomic ensemble. The inhomogeneity is primarily associated with a distribution of the atomic resonance frequencies leading to stimulated emission into the cavity at a range of different frequencies. Similar differences in the atom-field coupling due to variation in the atomic positioning are also included in the system.

We numerically investigate the dynamics of an atomic medium with a wide range of resonance frequencies and show how the intensity of the pumping rate can lead to cooperative effects among the atoms such that superradiant lasing is achieved. Furthermore, we consider atoms to have different coupling strengths to the cavity. We also study the laser sensitivity to cavity noise.

Model

We consider an ensemble of N incoherently pumped two-level atoms inside a single mode optical cavity as shown in Figure 1. In a bad-cavity regime, where the cavity relaxation rate exceeds the natural linewidth of the atomic transition by many orders of magnitude ($\kappa \gg \Gamma$), the system constitutes a generic model of a superradiant laser. The i-th atom couples to the cavity field with the coupling strength g_i and has a resonance frequency ω_i which might be shifted from the unperturbed atomic transition frequency ω_a. Assuming that the cavity is on resonance with the unperturbed atomic transition frequency, we describe the coherent dynamics of the system by the Tavis-Cummings Hamiltonian in the rotating frame of the cavity,

$$H = -\sum_{i=1}^{N} \Delta_i \sigma_i^+ \sigma_i^- + \sum_{i=1}^{N} g_i (a\sigma_i^+ + a^\dagger \sigma_i^-). \tag{1}$$

Here, $\Delta_i = \omega_c - \omega_i$, $\sigma_i^+ = (\sigma_i^-)^\dagger = |e\rangle_i\langle g|_i$ denote the raising and lowering operators of the i-th atom, where $|g\rangle$ and $|e\rangle$ are the atomic ground and excited states, respectively, and $a^\dagger$ (a) is the photon creation (annihilation) operator of the cavity mode. The dissipative processes of this system are described by the Liouvillian terms

$$\begin{aligned}
\mathscr{L}_\kappa[\rho] &= \frac{\kappa}{2}(2a\rho a^\dagger - a^\dagger a\rho - \rho a^\dagger a) \\
\mathscr{L}_\Gamma[\rho] &= \frac{\Gamma}{2}\sum_i (2\sigma_i^-\rho\sigma_i^+ - \sigma_i^+\sigma_i^-\rho - \rho\sigma_i^+\sigma_i^-) \\
\mathscr{L}_R[\rho] &= \frac{R}{2}\sum_i (2\sigma_i^+\rho\sigma_i^- - \sigma_i^-\sigma_i^+\rho - \rho\sigma_i^-\sigma_i^+),
\end{aligned} \tag{2}$$

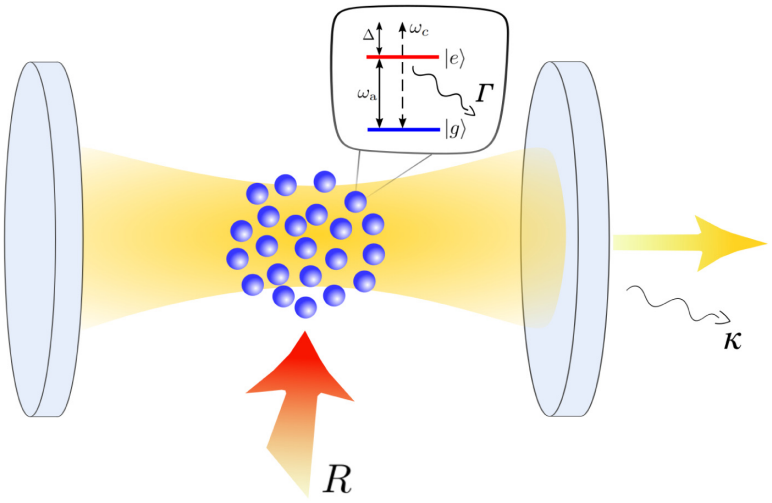


Figure 1. Schematic illustration of the system. The atomic medium is placed inside the optical resonator which has a resonance frequency ω_c. Each atom features a ground and an excited state separated by the transition energy ω_a. The transition couples to the cavity mode (g) as well as the environment (Γ). Additionally, the atoms are incoherently driven from the side (R) such that they can provide gain to the cavity mode.

representing the loss of photons through the cavity at the rate κ, the spontaneous atomic decay with the single-atom spontaneous emission rate Γ, and the individual incoherent pumping with the pump strength R. Thus, the full dynamics of the system is determined by the master equation for the density matrix ρ in standard Lindblad form

$$\dot{\rho} = -i[H,\rho] + \mathscr{L}_\kappa[\rho] + \mathscr{L}_\Gamma[\rho] + \mathscr{L}_R[\rho]. \tag{3}$$

Since the exponential growth of the Hilbert space with the number of atoms renders the solution of the master Equation 3 intractable for $N \gg 1$, we use a cumulant expansion method[7,28]. First, we write down the equations for operator averages describing our system, which for a given operator $\mathscr{O}$ reads

$$\begin{aligned}\frac{d}{dt}\langle\mathscr{O}\rangle = i\langle[H,\mathscr{O}]\rangle + \kappa\langle\mathscr{D}[a]\mathscr{O}\rangle + \\ \Gamma\sum_i\langle\mathscr{D}[\sigma_i^-]\mathscr{O}\rangle + R\sum_i\langle\mathscr{D}[\sigma_i^+]\mathscr{O}\rangle,\end{aligned} \tag{4}$$

where $\mathscr{D}[c]\mathscr{O} = (2c^\dagger\mathscr{O}c - c^\dagger c\mathscr{O} - \mathscr{O}c^\dagger c)/2$. We note that in some cases (mentioned in the description of the results) we additionally include cavity dephasing and atomic dephasing described by the terms $\xi\ \langle\mathscr{D}[a^\dagger a]\mathscr{O}\rangle$ and $\nu\sum_i\langle\mathscr{D}[\sigma_i^+\sigma_i^-]\mathscr{O}\rangle$, respectively. The cavity dephasing accounts for the effective noise imposed on the system by thermal fluctuations of the cavity mirrors, whereas the atomic dephasing models perturbations on the lasing transition.

To obtain a closed set of differential equations we use the cumulant expansion method[28] up to second order:

$$\begin{aligned}
&\frac{d}{dt}\langle a^\dagger a\rangle = -\kappa\langle a^\dagger a\rangle + i\sum_{m=1}^N g_m\langle a\sigma_m^+\rangle - i\sum_{m=1}^N g_m\langle a^\dagger\sigma_m^-\rangle \\
&\frac{d}{dt}\langle a\sigma_m^+\rangle = -((\kappa+\Gamma+R+\xi+\nu)/2 + i\Delta_m)\langle a\sigma_m^+\rangle + ig_m\langle a^\dagger a\rangle - 2ig_m\langle a^\dagger a\rangle\langle\sigma_m^+\sigma_m^-\rangle - ig_m\langle\sigma_m^+\sigma_m^-\rangle - i\sum_{j;m\neq j}^N g_j\langle\sigma_m^+\sigma_j^-\rangle \\
&\frac{d}{dt}\langle\sigma_m^+\sigma_m^-\rangle = ig_m\langle a^\dagger\sigma_m^-\rangle - ig_m\langle a\sigma_m^+\rangle - (\Gamma+R)\langle\sigma_m^+\sigma_m^-\rangle + R \\
&\frac{d}{dt}\langle\sigma_m^+\sigma_j^-\rangle = ig_m\langle a^\dagger\sigma_j^-\rangle - ig_j\langle a\sigma_m^+\rangle - 2ig_m\langle a^\dagger\sigma_j^-\rangle\langle\sigma_m^+\sigma_m^-\rangle + 2ig_j\langle a\sigma_m^+\rangle\langle\sigma_j^+\sigma_j^-\rangle - (\Gamma+R+\nu)\langle\sigma_m^+\sigma_j^-\rangle.
\end{aligned} \tag{5}$$

In order to calculate the spectrum of the cavity light field we make use of the Wiener–Khinchin theorem[29], which states that the spectrum can be computed as the Fourier transform of the first-order correlation function $g^{(1)}(\tau) = \langle a^\dagger(\tau)a(0)\rangle$,

$$S(\omega) = 2\Re\left\{\int_0^\infty d\tau e^{-i\omega\tau} g^{(1)}(\tau)\right\}. \tag{6}$$

We use the quantum regression theorem[30] to write down the set of differential equations for the two-time correlation function, which in matrix form reads

$$\frac{d}{d\tau}\begin{pmatrix} \langle a^\dagger(\tau)a(0)\rangle \\ \langle \sigma_1^+(\tau)a(0)\rangle \\ \vdots \\ \langle \sigma_N^+(\tau)a(0)\rangle \end{pmatrix} = \mathrm{A}\begin{pmatrix} \langle a^\dagger(\tau)a(0)\rangle \\ \langle \sigma_1^+(\tau)a(0)\rangle \\ \vdots \\ \langle \sigma_N^+(\tau)a(0)\rangle \end{pmatrix}, \tag{7}$$

where

$$\mathrm{A} = -\begin{pmatrix} \frac{\kappa+\xi}{2} & -ig_1 & \cdots & -ig_N \\ 2ig_1(\langle\sigma_1^+\sigma_1^-\rangle^{st} - \frac{1}{2}) & \frac{\Gamma+R+\nu}{2} + i\Delta_1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 2ig_N(\langle\sigma_N^+\sigma_N^-\rangle^{st} - \frac{1}{2}) & 0 & \cdots & \frac{\Gamma+R+\nu}{2} + i\Delta_N \end{pmatrix}. \tag{8}$$

We obtain the laser emission spectrum by taking the Laplace transform of Equation 7, where the initial conditions are the steady-state solutions of Equation 5, for example $\langle a^\dagger(\tau = 0)a(0)\rangle = \langle a^\dagger a\rangle^{st}$.

In this section, we suppose that all atoms in the ensemble are identical with the same detunings $\{\Delta_i\} = \Delta$ and couplings $\{g_i\} = g$ to the cavity mode. This reduces the problem to a set of four differential equations in Equation 5. The mean intra-cavity photon number and the laser linewidth $\Delta\nu$ (the FWHM of the spectrum) are depicted in Figure 2 as function of the number of atoms, pumping rate, and atom-cavity coupling strength.

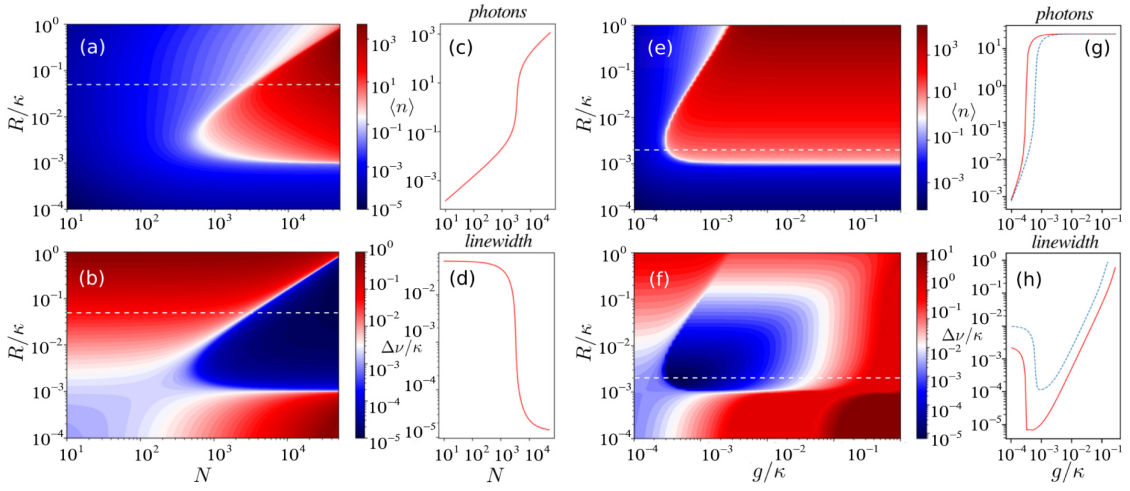


Figure 2. (**a**) The mean photon number and (**b**) the linewidth (in units of κ) as functions of the number of atoms N and pumping rate R for the parameter set $(\Delta, g, \Gamma, \xi, \nu) = (0, 0.002\kappa, 0.001\kappa, 0, 0)$. (**c**-**d**) The cut through the white dashed line in (**a**-**b**) for $R = 0.05\kappa$. (**e**-**f**) The mean photon number and the linewidth as functions of the atom-cavity coupling strength g and pumping rate R. Additional cavity dephasing occurs at the rate $\xi = \kappa$. Parameters: $\Delta = 0$, $\Gamma = 0.001\kappa$, $N = 5 \times 10^4$. (**g**-**h**) The cut through the white dashed line in (**e**-**f**): the ultra-narrow linewidth is robust to cavity dephasing $\xi = \kappa$ (red solid line) in the regime where the photon number is low. For the blue dashed line atomic dephasing was added to the system with the rate $\nu = 10\Gamma$.

Superradiance is expected in the parameter regime where the single-atom cooperativity parameter $C = 4g^2/(\kappa\Gamma) < 1$, but the system is in the collective strong coupling regime[7], where $CN \gg 1$. Figure 2(a-d) show the emergence of the superradiant regime as the number of atoms increases. Above the lasing threshold the collective emission of light with an ultra-narrow linewidth is observed. In this collective regime the phases of the atomic dipoles are synchronized via photon exchange through the cavity which leads to the buildup of a collective dipole among the atoms.

A key feature of such a laser is its insensitivity to thermal and mechanical fluctuations of the cavity length, since the coherence is primarily stored in the atoms rather than in the cavity field. To show the robustness against cavity noise we include cavity dephasing with the rate ξ in the equations. In Figure 2(f) we scan the linewidth over the coupling strength g and pumping rate R for an ensemble of $N = 5 \times 10^4$ atoms. In the superradiant regime, the laser linewidth is less than the natural linewidth of the atomic transition and approaches the value $\Delta\nu \sim C\Gamma$, which can be well below 1 mHz for the $^1S_0 \rightarrow {}^3P_0$ transition in ^{87}Sr, as has been pointed out in Ref. 7. Furthermore, we study the influence of noise on the laser linewidth in more detail. In Figure 3 we scan the linewidth over both cavity and atomic dephasing, where the other parameters of the system correspond to the superradiant regime. One can see that the linewidth of the superradiant laser can be extremely robust to noise sources within a wide range.

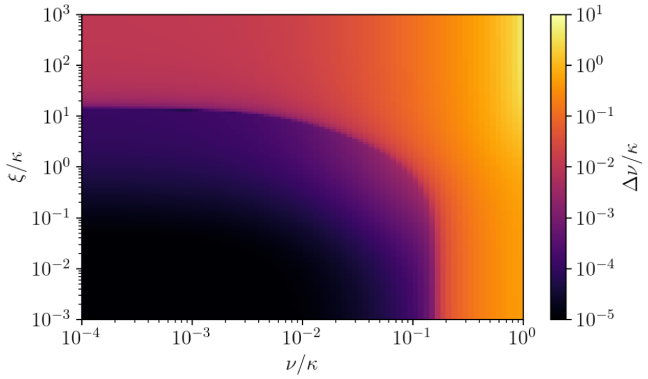


Figure 3. The linewidth of the emission spectrum of $N = 5{\times}10^4$ atoms as a scan over cavity dephasing (ξ) and atomic dephasing (ν). The optimal parameters are taken from Figure 2(f), where the system is in the superradiant regime for $(\Delta, g, \Gamma, R) = (0, 0.001\kappa, 0.001\kappa, 0.01\kappa)$.

So far the results are based on the idea of absolutely identical atoms. In the next sections, we focus on inhomogeneity within the atomic medium. In particular, we will consider the atoms to be subject to distinct frequency shifts and different couplings to the resonator mode.

Atomic ensembles with inhomogeneous broadening

While the individual atoms in free space are identical and have the same transition frequencies in principle, in practise they are often subject to individual perturbations introducing local lineshifts, e.g. from trapping within the cavity, motion, or optical pumping. Specifically, it can be an inhomogeneous trapping lattice or pump lasers with a Gaussian profile. Doppler shifts would have similar broadening effects in ring cavities, whereas in a standing-wave cavity they would generate a time-dependent atom-field coupling which we do not consider here. In this section we study the overall effects of inhomogeneous broadening of the gain medium on the laser properties.

In contrast to the case of identical atoms, where the atom number in Equation 5 and Equation 7 only enters as a constant factor, the inhomogeneity among atomic frequencies requires keeping track of the time evolution of each atom separately. For the solution of the collective dynamics one then needs to solve $\mathcal{O}(N^2)$ equations. This is only possible for a limited atom number and we thus have to resort to further approximation methods in order to treat larger ensembles. As a possible approach to approximate a large ensemble with a continuous frequency distribution we combine several atoms in subgroups representing their average atomic frequencies, which we call clusters, see also Refs. 14,15,31. Each atom in a cluster is assumed to be completely identical to all other

atoms in the same frequency cluster. This preserves the central physics of the inhomogeneous broadening, but at the same time substantially reduces the number of equations.

First, we simulate $N = 5$ atoms in five clusters centered at $\Delta_m = \omega_c - \omega_m$, where $\Delta_m \in [-\kappa : \kappa]$. Note that this is equivalent to $M = 5$ frequency clusters each containing a single atom. At low excitation the resulting cavity output spectrum then consists of precisely five spectral lines at the frequency of each cluster. Basically, these are five independent lasers using the same cavity mode simultaneously. If we increase N and set the number of atoms per cluster according to a Gaussian normal distribution with the standard deviation $\sigma = \kappa$, the structure of the spectrum in Figure 4(a) will remain unchanged, with each peak becoming more pronounced. In particular, in Figure 4(b) we observe growing collective emission among atoms of the same cluster so that the linewidth of each peak becomes smaller as the atom number in the corresponding cluster increases. In Figure 4(c) we show how more and more lines appear as we increase the number of clusters up to $M = 201$ until the output merges into a single broad emission line. Note that an increase of the collective coupling to $g\sqrt{N} \sim \kappa$ or a randomization of the individual cluster detunings do not lead to any substantial difference in the spectral profile of the laser. Hence, one can expect a single broadened peak in the emission spectrum in the more realistic case of a large ensemble of atoms with a continuous frequency distribution.

So far we limited investigations to weak incoherent pumping in order to avoid significant additional broadening of the atomic linewidth due to pumping. However, this broadening effect can actually aid the buildup of coherences between the clusters. When the pumping is strong enough such that the distinct spectral lines overlap, the discrete spectral lines of the clusters merge into a single central peak (see Figure 5). In other words, more intra-cavity photons and broader individual atomic gain lines ultimately lead to a dramatic narrowing of the laser line. We attribute this effect to a dynamical phase transition from the unsynchronized phase of the dipoles to the synchronized one. Note that an analogous phenomenon has previously been studied in Ref. 32 for two mesoscopic ensembles of atoms collectively coupled to a cavity with opposite detunings. Furthermore, we show how an atom number imbalance at a particular frequency in Figure 5(b) and overall atom number fluctuations modeled by slight random deviations from a Gaussian distribution in Figure 5(c) lead to a shift of the spectral lines. However, in the synchronized regime the lineshift of the central peak is much smaller than its linewidth.

The collapse of the emission spectrum into a single central line occurs at a critical pump strength R_c. This critical value strongly depends on the overall width of the frequency distribution, but shows almost no dependence on the number of subensembles M and the total number of atoms N. The critical transition pump strength is shown for different standard deviations σ of the atomic frequency distribution in Figure 6. The data points show the numerical results for an ensemble of $N = 10^2$ (red dots) and $N = 10^4$ (blue circles) atoms sampled by $M = 31$ clusters. For comparison, we also plot the linear (solid line) function $R_c = 0.4\sigma$. We calculate the critical pumping by computing the spectrum for different R. We then determine the critical value of the

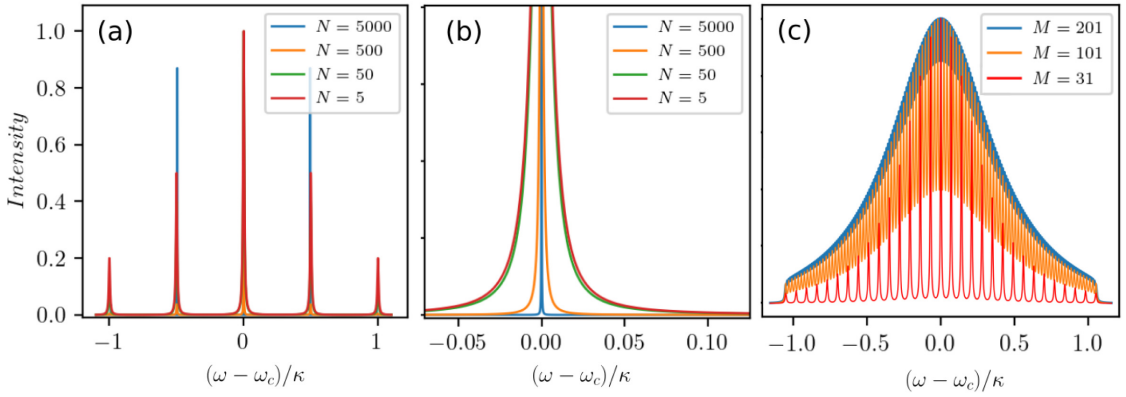


Figure 4. Cavity output spectra for weakly driven atomic ensembles composed of several discrete clusters with varying atomic frequencies. (**a**) $M = 5$ clusters of atoms with the detunings $\Delta_m = [-\kappa; -\kappa/2; 0; \kappa/2; \kappa]$ for different total numbers of atoms $N = 5, \ldots, 5000$. (**b**) A zoom-in showing the narrowing of the central peak in the spectrum from (**a**) around the resonance frequency. (**c**) Transition of the spectral distribution from discrete to quasi-continuous for an increasing number of clusters. Parameters: $(g, \Gamma, R) = (0.002\kappa; 0.001\kappa; 0.01\kappa)$.

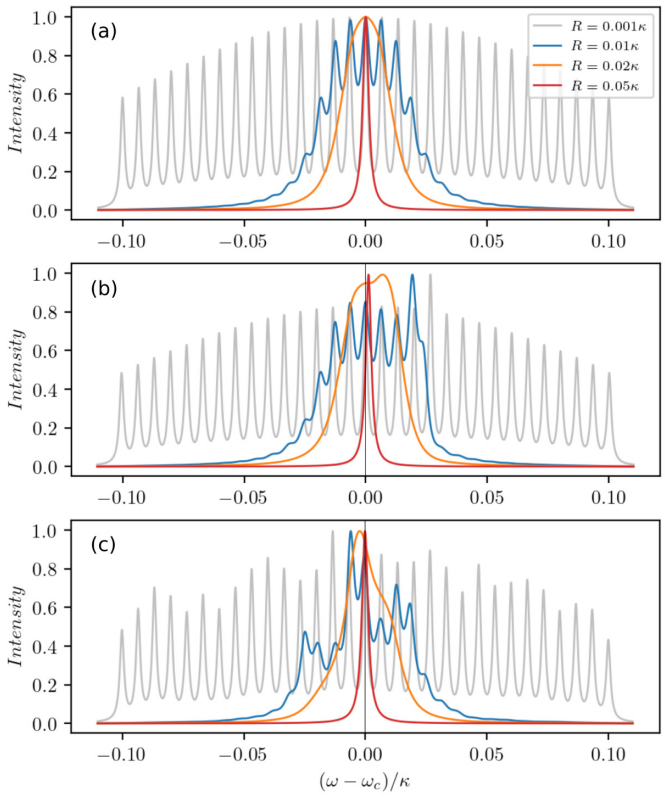


Figure 5. Cavity output spectra of a large inhomogeneously broadened ensemble of $N = 10^4$ atoms for different pumping rates $R = 0.001\kappa$ (grey), 0.01κ (blue), 0.02κ (orange), 0.05κ (red). The ensemble is represented by $M = 31$ clusters with the number of atoms per cluster chosen according to a Gaussian normal distribution (**a**) with the standard deviation $\sigma = 0.1\kappa$; (**b**) when adding particle imbalance at $\Delta = 0.027\kappa$; (**c**) with overall atom number fluctuations. The emission intensity is normalized and the other parameters are chosen as $\Delta \in [-\sigma : \sigma]$, $g = 0.002\kappa$, $\Gamma = 0.001\kappa$.

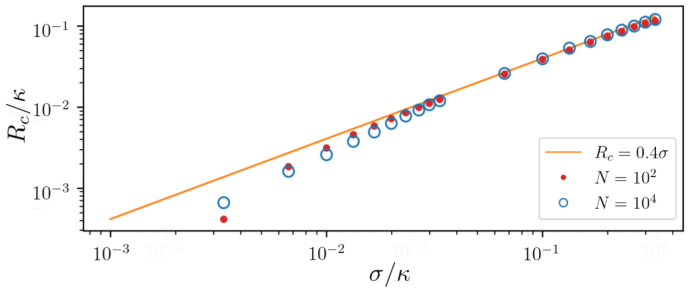


Figure 6. Critical value of pumping above which the collective superradiant regime is established depending on the standard deviation σ of the atomic frequency distribution. The data points show the numerical results for an ensemble of $N = 10^2$ (red dots) and $N = 10^4$ (blue circles) Open blue circles show the numerical results for the ensemble of $N = 10^4$ atoms sampled by $M = 31$ clusters. For comparison we plot the linear (solid line) function $R_c = 0.4\sigma$ as a linear approximation to the data points. Parameters: $\Delta \in [-3\sigma : 3\sigma]$, $g = 0.001\kappa$, $\Gamma = 0.001\kappa$.

pump strength as the value at which the spectrum has only a single local maximum, i.e. all separate peaks have merged into a single spectral line. We find a linear dependence for large inhomogeneously broadened ensembles while for narrow ensembles a significantly lower pump strength is required.

Once the laser is operating at a single distinct emission frequency, we can characterize the properties of the output light by the linewidth and the average photon number. The results for different distributions of atomic

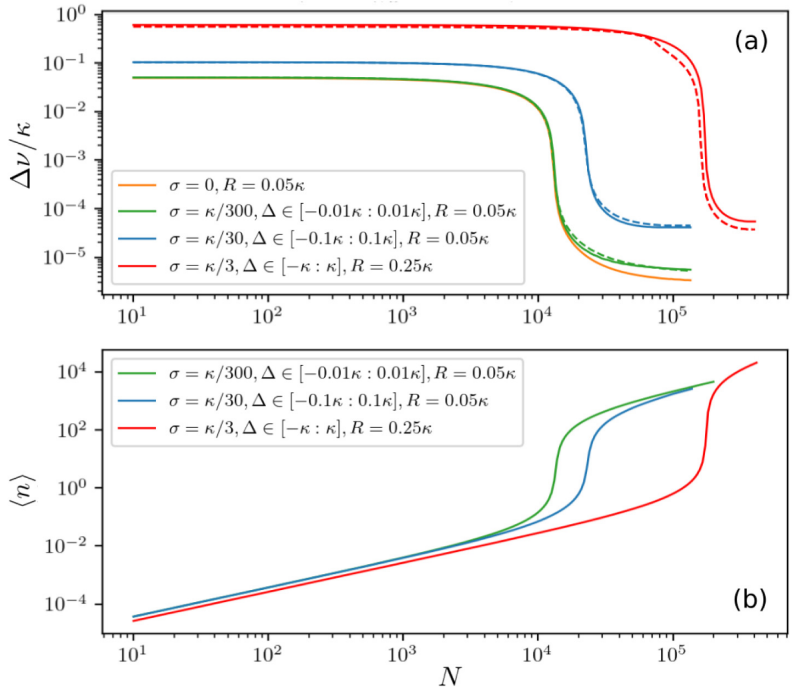


Figure 7. (**a**) Laser spectral linewidth and (**b**) mean photon number for inhomogeneously broadened ensembles with different standard deviations σ and spectral widths of atomic frequencies $\Delta \in [-3\sigma : 3\sigma]$, where $\sigma = \kappa/300$ (green line), $\sigma = \kappa/30$ (blue line), $\sigma = \kappa/3$ (red line) as a function of the total number of atoms. The number of clusters is $M = 31$ with the number of atoms per cluster chosen according to a Gaussian normal distribution. The dashed lines represent the results including an additional spatial variation of the atom-field coupling $g(x) = g_0 \cos(\kappa x)$. The ensemble is comprised of $M = 11$ frequency clusters and $K = 5$ clusters of different couplings. The couplings are chosen such that the effective coupling strength $g_{\text{eff}} = \sqrt{(\sum_m g_m^2)/K} \equiv g$. Parameters: $g = 0.001\kappa$; $g_0 = 0.0013\kappa$; $\Gamma = 0.001\kappa$.

frequencies are shown in Figure 7, where $\Delta \in [-3\sigma : 3\sigma]$ and $\Gamma \leq 3\sigma \leq \kappa$. Figure 7(a) illustrates how a narrow linewidth appears for different σ as the number of atoms increases. Note that we chose a pumping strength well above the critical value for a wide atomic frequency distribution (red line). The sharp decrease of the linewidth is accompanied by an increase in the average photon number as can be seen in Figure 7(b). This is indicative of a lasing threshold being crossed at a certain number of atoms.

Inhomogeneously broadened ensembles with variable coupling strength

Up to now we have assumed that the atoms are perfectly positioned inside the cavity such that they couple equally to the cavity mode. Let us now include spatial variations of the atom-field coupling within the resonator. We consider the ensemble of atoms with the position-dependent coupling strength $g(x) = g_0 \cos(kx)$, where g_0 is the coupling constant, $k = 2\pi/\lambda$ is the cavity mode wave number and x represents the position of an atom. In order to describe the atom-field dynamics we use a similar cluster approach as before. We assume equidistant positions for different clusters $x_m \in [0, ..., \lambda/4)$ and corresponding couplings $g_m(x) = g_0 \cos(kx_m) = \{g_1, g_2, ..., g_K\}$, where K is the total number of clusters. Note, that the sign of the coupling is irrelevant in our system, therefore we only consider couplings with $g_m > 0$.

The dashed lines in Figure 7 show the results for $M = 11$ frequency clusters and $K = 5$ clusters of different couplings. As can be seen in Figure 7(a), for atoms with different couplings to the cavity mode the dependence of the linewidth on the number of atoms remains roughly the same as for atoms equally coupled to the cavity. This holds as long as the effective overall coupling strength $g_{\text{eff}} = \sqrt{(\sum g_m^2)/K}$ is constant. Thus, the linewidth is essentially unaffected by atoms having different couplings to the cavity.

Finally, let us include cavity dephasing in order to describe lasing in a large inhomogeneously broadened ensemble in the presence of cavity noise. The spectral linewidth and mean photon number under strong cavity dephasing at the rate $\xi = \kappa$ are depicted in Figure 8 (blue dashed line). Note that establishing coherence in such a

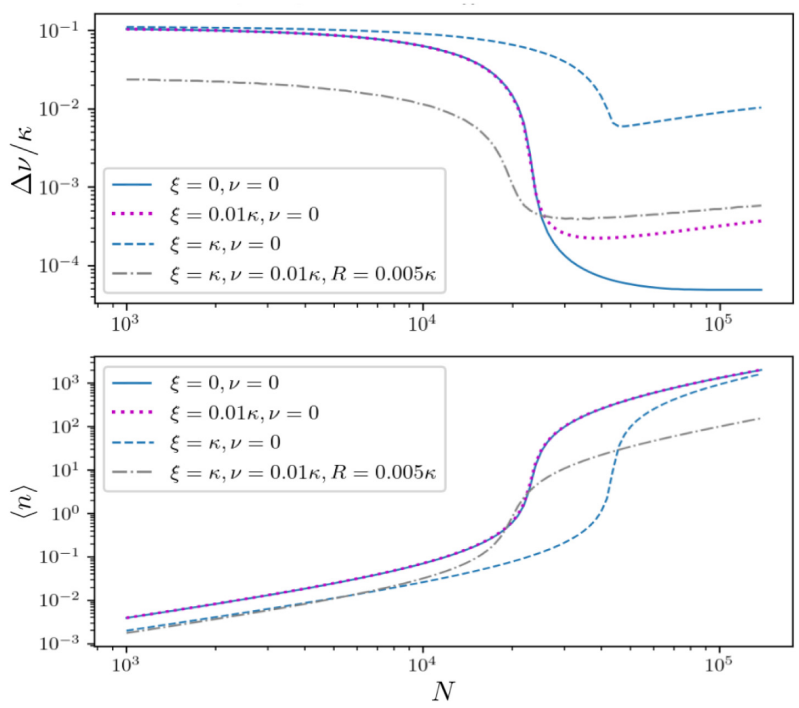


Figure 8. Laser linewidth (upper panel) and mean photon number (lower panel) for an inhomogeneously broadened ensemble with spatially varying coupling for $\sigma = \kappa/30$, $\Delta \in [-0.1\kappa : 0.1\kappa]$ and $R = 0.05\kappa$ (solid line). Adding various cavity dephasing at the rate $\xi = \kappa$ (dashed blue line) and $\xi = 0.01\kappa$ (dotted magenta line) we can identify an optimal atom number, above which the cavity noise overwhelms the linewidth narrowing due to large photon numbers. The dash-dotted grey line shows the results when adding additional atomic dephasing at the rate $\nu = 0.01\kappa$. This additional broadening allows synchronization of the individual clusters in the weak pumping regime $R = 0.005\kappa$ ultimately leading to a smaller linewidth.

largely broadened ensemble requires sufficiently strong pumping. This subsequently leads to a large number of photons in the cavity mode making the setup sensitive to cavity fluctuations, see Figure 2(f). However, additional atomic dephasing can actually relax the constraint on the pumping, since both incoherent pumping and atomic dephasing are closely tied to the same physical effect of broadening the atomic emission line. Thus individual atomic dephasing induce additional atom-atom coupling by enlarging the overlap of distinct spectral lines, which finally leads to better synchronization. Adding atomic dephasing to the system at the rate $\nu = 0.01\kappa$ allows for maintaining collective interactions in the ensemble and at the same time enables a reduction of the pump strength by one order of magnitude to $R = 0.005\kappa$. In the low photon number regime, a linewidth on the order of the natural atomic linewidth Γ can be achieved in the presence of strong atomic and cavity dephasing (dash-dotted grey line).

The presented results can be reproduced by using the source code *N_atoms_M_clusters_Delta.jl* (see *Software availability*)[33]. The file contains an example of the cluster approach written in Julia version 1.5.0 using the parameters in Figure 5. Numerical simulations were performed with the open-source framework Differentialequations.jl[34]. The toolbox QuantumCumulants.jl[35] has been used to check the equations and verify the second-order cumulant expansion. The graphs were produced using the Matplotlib library[36].

Conclusions

We studied superradiant lasing when the gain medium is subject to substantial inhomogeneous frequency broadening and variable coupling. In extensive numerical simulations based on a second-order cumulant expansion we were able to confirm previous predictions that sufficiently large numbers of atoms subject to strong optical pumping can induce synchronization of the atomic dipoles over a large bandwidth. This generates collective stimulated emission of light into the cavity mode leading to narrow-band laser emission at the

average of the atomic frequency distribution. The linewidth is orders of magnitudes smaller than that of the cavity as well as the inhomogeneous gain broadening and exhibits reduced sensitivity to cavity frequency noise. We determine the operational conditions and, in particular, the best pump rate to choose for achieving the smallest linewidth for a given atom number and cavity. The minimum occurs not at very low photon numbers but at intra-cavity photon numbers reaching a significant fraction of the atom number.

Typically, full synchronization requires fairly strong pumping, which increases the effective atomic linewidth. We determined the minimum pump strength to achieve collective phase-locked oscillation of all atomic dipoles. Interestingly, some individual line-broadening effects such as atomic dephasing can actually induce synchronization at significantly lower pump rates. Furthermore, our simulations also show that variations in the atom-field coupling strength induced by the cavity mode structure play only a minor role for the laser stability and noise. In fact, they can be compensated by an increase of the effective overall coupling using a larger atom number or stronger pump.

In the present work, we did not take into account collisions or dipole-dipole interactions between atoms. The effect of dipole-dipole interactions have been studied in a small-scale full quantum model in Ref. 9 and do not appear too detrimental. Moreover, collisions could even have a positive effect on synchronization[37] but a quantitative prediction is complicated. So far our model is still based on a very simplistic effective pump description via an individual, independent and equal pump rate for each atom. More detailed studies of optical pumping schemes including the shifts induced by the pump light will be at the center of future studies.

Appendix. Cross-correlations between atoms in different clusters

As we refer to in the main text, we model a continuous atomic frequency distribution with the standard deviation σ by choosing equidistant cluster detunings Δ_m with the number of atoms per cluster N_m given by a Gaussian distribution with the standard deviation σ. The Heisenberg equations for an ensemble of N atoms sampled by M clusters can be written as

$$\frac{d}{dt}\langle a^\dagger a\rangle = -\kappa\langle a^\dagger a\rangle + ig\sum_{m=1}^{M} N_m\langle a\sigma_m^+\rangle - ig\sum_{m=1}^{M} N_m\langle a^\dagger\sigma_m^-\rangle$$

$$\frac{d}{dt}\langle a\sigma_m^+\rangle = -((\kappa+\Gamma+R)/2 + i\Delta_m)\langle a\sigma_m^+\rangle + ig\langle a^\dagger a\rangle - 2ig\langle a^\dagger a\rangle\langle\sigma_{am}^+\sigma_{am}^-\rangle - ig\langle\sigma_{am}^+\sigma_{am}^-\rangle + \ldots$$

$$\ldots - ig(N_m-1)\langle\sigma_{am}^+\sigma_{bm}^-\rangle - ig\sum_{j;m\neq j}^{M} N_j\langle\sigma_m^+\sigma_j^-\rangle$$

$$\frac{d}{dt}\langle\sigma_{am}^+\sigma_{am}^-\rangle = ig\langle a^\dagger\sigma_m^-\rangle - ig\langle a\sigma_m^+\rangle - (\Gamma+R)\langle\sigma_{am}^+\sigma_{am}^-\rangle + R$$

$$\frac{d}{dt}\langle\sigma_{am}^+\sigma_{bm}^-\rangle|_{a\neq b} = ig\langle a^\dagger\sigma_m^-\rangle - ig\langle a\sigma_m^+\rangle - 2ig\langle a^\dagger\sigma_m^-\rangle\langle\sigma_{am}^+\sigma_{am}^-\rangle + 2ig\langle a\sigma_m^+\rangle\langle\sigma_{am}^+\sigma_{am}^-\rangle - (\Gamma+R)\langle\sigma_{am}^+\sigma_{bm}^-\rangle.$$

$$\frac{d}{dt}\langle\sigma_m^+\sigma_j^-\rangle|_{m\neq j} = -i(\Delta_m-\Delta_j)\langle\sigma_m^+\sigma_j^-\rangle + ig\langle a^\dagger\sigma_j^-\rangle(1-2\langle\sigma_{am}^+\sigma_{am}^-\rangle) - ig\langle a\sigma_m^+\rangle(1-2\langle\sigma_{aj}^+\sigma_{aj}^-\rangle) - (\Gamma+R)\langle\sigma_m^+\sigma_j^-\rangle, \quad \text{(A1)}$$

where indices a, b refer to an atom, and m, j are cluster indices. The last equation describes the cross-correlations between atoms in different clusters. Next, we study the phase and the amplitude of these correlations as the system reaches the steady-state. In the weak pumping regime, the correlations are zero and therefore there is no coherence between the distinct spectral lines of the output spectra in Figure 4. However, in the synchronized regime shown in Figure 5(a) for $R = 0.05\kappa$, the existing cross-correlations of the m-th cluster with the other clusters $j = m..M$ are presented in Figure 9(a).

Let us follow these correlations as the system goes from the unsynchronized phase to the synchronized one. We study the magnitude of cross-correlations between the first (outer) cluster and the central cluster in Figure 5(a) as a function of the pumping strength. The correlations are zero in the weak pumping regime and grow with the pumping strength as shown in Figure 9(b). The function reaches its maximal value when the ensemble is fully synchronized. However, as pumping continues to grow the correlations decrease due-to growing dephasing imposed by pumping.

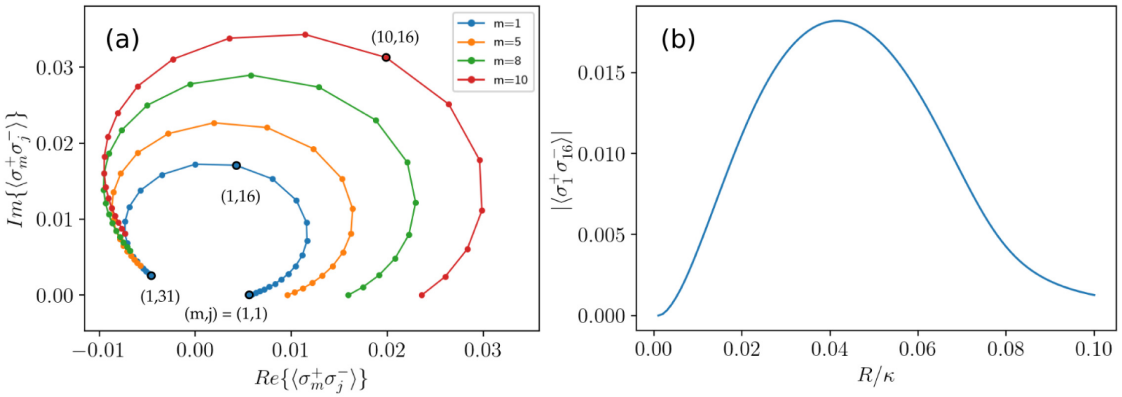


Figure 9. Cross-correlations between the 31 clusters presented in Figure 5(a). (**a**) Real and imaginary part of $\langle\sigma_m^+\sigma_j^-\rangle$ correlations between atoms in the *m*-th and *j*-th clusters on the complex plane for $R = 0.05\kappa$. (**b**) The magnitude of the cross-correlations between atoms in the first and the central clusters as a function of the pumping strength.

Data availability

Underlying data

Figshare: Superradiant_laser_Figures. https://doi.org/10.6084/m9.figshare.15321819[38].

This project contains the following underlying data:

- Data used in Figures 2–8. All data have .jld2 file extension using JLD2.jl data package in Julia.

Data are available under the terms of the Creative Commons Zero "No rights reserved" data waiver (CC0 1.0 Public domain dedication).

Software availability

- Source code available from: https://github.com/by-anna/Clusters
- Archived source code at time of publication: https://doi.org/10.5281/zenodo.4916393[33]
- License: MIT License

References

1. Dicke RH: **Coherence in spontaneous radiation processes.** *Phys Rev.* 1954; **93**(1): 99.
Publisher Full Text
2. Bonifacio R, Schwendimann P, Haake F: **Quantum statistical theory of superradiance. i.** *Phys Rev A.* 1971; **4**(1): 302.
Publisher Full Text
3. Haake F, Kolobov MI, Fabre C, *et al.*: **Superradiant laser.** *Phys Rev Lett.* 1993; **71**(7): 995.
PubMed Abstract | **Publisher Full Text**
4. Benedict MG: **Super-radiance:Multiatomic Coherent Emission**. CRC Press, 1996.
Reference Source
5. Scully MO, Svidzinsky AA: **Physics. The super of superradiance.** *Science.* 2009; **325**(5947): 1510–1511.
PubMed Abstract | **Publisher Full Text**
6. Chen J: **Active optical clock.** *Chin Sci Bull.* 2009; **54**(3): 348–352.
Publisher Full Text
7. Meiser D, Ye J, Carlson D, *et al.*: **Prospects for a millihertz-linewidth laser.** *Phys Rev Lett.* 2009; **102**(16): 163601.
PubMed Abstract | **Publisher Full Text**
8. Meiser D, Holland M: **Steady-state superradiance with alkaline-earth-metal atoms.** *Phys Rev A.* 2010; **81**(3): 033847.
Publisher Full Text
9. Maier T, Kraemer S, Ostermann L, *et al.*: **A superradiant clock laser on a magic wavelength optical lattice.** *Opt Express.* 2014; **22**(11): 13269–13279.
PubMed Abstract | **Publisher Full Text**
10. Zhang Y, Zhang YX, Mølmer K: **Monte-carlo simulations of superradiant lasing.** *New J Phys.* 2018; **20**(11): 112001.
Publisher Full Text
11. Debnath K, Zhang Y, Mølmer K: **Lasing in the superradiant crossover regime.** *Phys Rev A.* 2018; **98**(6): 063837.
Publisher Full Text
12. Hotter C, Plankensteiner D, Ostermann L, *et al.*: **Superradiant**

cooling, trapping, and lasing of dipole-interacting clock atoms. *Opt Express.* 2019; **27**(22): 31193–31206.
PubMed Abstract | Publisher Full Text

13. Gogyan A, Kazakov G, Bober M, *et al.*: **Characterisation and feasibility study for superradiant lasing in 40 ca atoms.** *Opt Express.* 2020; **28**(5): 6881–6892.
PubMed Abstract | Publisher Full Text

14. Shankar A, Reilly JT, Jäger SB, *et al.*: **Subradiant-to-subradiant phase transition in the bad cavity laser.** *arXiv preprint arXiv: 2103.07402.* 2021.
Reference Source

15. Wu Q, Zhang Y, Yang X, *et al.*: **A superradiant maser with nitrogen-vacancy center spins.** *arXiv preprint arXiv: 2105.12350*. 2021.
Reference Source

16. Bohnet JG, Chen Z, Weiner JM, *et al.*: **A steady-state superradiant laser with less than one intracavity photon.** *Nature.* 2012; **484**(7392): 78–81.
PubMed Abstract | Publisher Full Text

17. Norcia MA, Thompson JK: **Cold-strontium laser in the superradiant crossover regime.** *Phys Rev X.* 2016; **6**(1): 011025.
Publisher Full Text

18. Norcia MA, Winchester MN, Cline JR, *et al.*: **Superradiance on the millihertz linewidth strontium clock transition.** *Sci Adv.* 2016; **2**(10): e1601231.
PubMed Abstract | Publisher Full Text | Free Full Text

19. Chen CC, Bennetts S, Escudero RG, *et al.*: **Continuous guided strontium beam with high phase-space density.** *Phys Rev Applied.* 2019; **12**(4): 044014.
Publisher Full Text

20. Laske T, Winter H, Hemmerich A: **Pulse delay time statistics in a superradiant laser with calcium atoms.** *Phys Rev Lett.* 2019; **123**(10): 103601.
PubMed Abstract | Publisher Full Text

21. Schäffer SA, Tang M, Henriksen MR, *et al.*: **Lasing on a narrow transition in a cold thermal strontium ensemble.** *Phys Rev A.* 2020; **101**(1): 013819.
Publisher Full Text

22. Tang M, Schäffer SA, Jørgensen AA, *et al.*: **Cavity-immune features in the spectra of superradiant crossover laser pulses.** *arXiv preprint arXiv: 2104.13305.* 2021.
Reference Source

23. Numata K, Kemery A, Camp J: **Thermal-noise limit in the frequency stabilization of lasers with rigid cavities.** *Phys Rev Lett.* 2004; **93**(25): 250602.
PubMed Abstract | Publisher Full Text

24. Notcutt M, Ma LS, Ludlow AD, *et al.*: **Contribution of thermal noise to frequency stability of rigid optical cavity via hertz-linewidth lasers.** *Phys Rev A.* 2006; **73**(3): 031804.
Publisher Full Text

25. Liu H, Jäger SB, Yu X, *et al.*: **Rugged mhz-linewidth superradiant laser driven by a hot atomic beam.** *Phys Rev Lett.* 2020; **125**(25): 253602.
PubMed Abstract | Publisher Full Text

26. Jäger SB, Liu H, Shankar A, *et al.*: **Regular and bistable steady-state superradiant phases of an atomic beam traversing an optical cavity.** *Phys Rev A.* 2021; **103**(1): 013720.
Publisher Full Text

27. Norcia MA, Young AW, Eckner WJ, *et al.*: **Seconds-scale coherence in a tweezer-array optical clock.** *arXiv preprint arXiv: 1904.10934.* 2019.
Reference Source

28. Kubo R: **Generalized cumulant expansion method.** *J Phys Soc Jpn.* 1962; **17**(7): 1100–1120.
Publisher Full Text

29. Puri RR: **Mathematical methods of quantum optics.** Springer Science & Business Media, 2001; 79.
Publisher Full Text

30. Carmichael HJ: **Statistical methods in quantum optics 1: master equations and Fokker-Planck equations.** Springer Science & Business Media, 2013.
Reference Source

31. Debnath K, Zhang Y, Mølmer K: **Collective dynamics of inhomogeneously broadened emitters coupled to an optical cavity with narrow linewidth.** *Phys Rev A.* 2019; **100**(5): 053821.
Publisher Full Text

32. Xu M, Tieri DA, Fine EC, *et al.*: **Synchronization of two ensembles of atoms.** *Phys Rev Lett.* 2014; **113**(15): 154101.
PubMed Abstract | Publisher Full Text

33. Bychek A: **Clusters.** *Zenodo.* 2021.
http://www.doi.org/10.5281/zenodo.4916393

34. Rackauckas C, Nie Q: **Differentialequations.jl – a performant and feature-rich ecosystem for solving differential equations in julia.** *J Open Res Softw.* 2017; **5**(1): 15.
Publisher Full Text

35. Plankensteiner D, Hotter C, Ritsch H: **Quantumcumulants.jl: A julia framework for generalized mean-field equations in open quantum systems.** *arXiv preprint arXiv: 2105.01657.* 2021.
Reference Source

36. Hunter JD: **Matplotlib: A 2D graphics environment.** *Comput Sci Eng.* 2007; **9**(3): 90–95.
Publisher Full Text

37. Zhu B, Schachenmayer J, Xu M, *et al.*: **Synchronization of interacting quantum dipoles.** *New J Phys.* 2015; **17**(8): 083063.
Publisher Full Text

38. Bychek A: **Superradiant_laser_figures.** *Figshare.* 2021.
http://www.doi.org/10.6084/m9.figshare.15321819.v1

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Current Peer Review Status: ✔ ✔

Version 2

Reviewer Report 04 October 2021

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✔ **Kamanasish Debnath**
Wallenberg Centre for Quantum Technology, Chalmers University of Technology, Gothenburg, Sweden

The authors have addressed all my comments and I would be happy to approve the article.

Competing Interests: No competing interests were disclosed.

Reviewer Expertise: Quantum optics

I confirm that I have read this submission and believe that I have an appropriate level of expertise to confirm that it is of an acceptable scientific standard.

Reviewer Report 27 September 2021

https://doi.org/10.21956/openreseurope.15259.r27659

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✔ **David Petrosyan**
Institute of Electronic Structure & Laser, Foundation for Research and Technology-Hellas, GR-70013 Heraklion, Crete, Greece

The authors addressed all the comments and questions clearly and convincingly. I have no further comments on this manuscript.

Competing Interests: No competing interests were disclosed.

Reviewer Expertise: Quantum optics

I confirm that I have read this submission and believe that I have an appropriate level of expertise to confirm that it is of an acceptable scientific standard.

Version 1

Reviewer Report 07 July 2021

https://doi.org/10.21956/openreseurope.14860.r27159

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David Petrosyan
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[2] Institute of Electronic Structure & Laser, Foundation for Research and Technology-Hellas, GR-70013 Heraklion, Crete, Greece

The authors present the results of their numerical studies of an ensemble of atoms coupled to a common mode of an optical cavity. The atoms are modeled as two-level systems incoherently pumped to the excited state from where they can spontaneously decay to the ground state or coherently emit a photon into the cavity mode. To treat large number of atoms N, the authors uses the method of cumulant expansion up to second order in the field and atomic operators, which permits calculations of the first-order field correlation function, with its Fourier transform being the power spectrum of the emitted radiation.

It has been shown before that even when the cavity linewidth is much larger than the natural linewidth of the atomic transition, the collective atomic coherence can result in superradiant lasing with very narrow linewidth, many orders of magnitude smaller than that of an empty cavity. This can have many interesting and important applications in spectroscopy, precision measurement, optical clocks, etc. The authors first reproduce these results, and determine the appropriate values of the pumping rate R, the atom number N, and the atom-cavity coupling strength g (also taking into account the cavity dephasing rate ξ), corresponding to the collective strong coupling regime.

Then the authors study the superradiant lasing from inhomogeneous atomic ensembles, with different atoms having different transition frequencies, but still the same transition linewidth. For moderate pumping, each subensemble of atoms produces superradiant lasing with its own frequency and the total lasing linewidth is that of the inhomogeneous atomic linewidth. But for stronger pumping, a dynamical transition to narrow-line synchronized lasing can occur. The authors also consider spatial variations of the coupling strengths of the atoms to the cavity mode, and show that the superradiant lasing can still occur with the linewidth determined by an effective averages coupling strength g_{eff}. Finally, the authors show that detrimental effects of the cavity

dephasing can be significantly reduced by introducing additional atomic dephasing. This is a counterintuitive result and it would be useful if the authors can provide a physical interpretation for it.

Overall, this is a well-written manuscript presenting a good quality work, which I approve with minor recommendations:

- On page 6, in the third paragraph, it should be the Gaussian distribution of the atomic transition frequencies σ=κ (not atom number).
- In Fig. 4 perhaps it should be stated that what is shown is the normalized intensity (or max intensity is set to 1).
- In Fig. 5, the critical value of the pumping rate vs inhomogeneous linewidth σ is shown for $N=10^4$ atoms. How does it change with the atom number? Is it approximately inversely proportional to N, for otherwise the same parameters?
- The authors mention that inhomogeneous broadening of the atomic transition can originate from trapping within the cavity, motion, or optical pumping. It would be useful if the author could expand the discussion a bit and clarify these issues. Do they mean that, depending on the atom position in the trap, its transition frequency will be different? Or that atom motion leads to the Doppler frequency shift? How does optical pumping lead to different frequency shifts of different atoms?
- The present manuscript did not explicitly consider high-density atomic ensembles, where the atomic collisions or dipole-dipole interactions would lead to increased coherence relaxation or collective broadening of the atomic transition. Would the authors be in a position to give some informed speculations as to whether the superradiant lasing with narrow line spectrum will still be possible in such situations, compensating thereby the detrimental broadening due to the interatomic interactions?

Is the work clearly and accurately presented and does it cite the current literature?
Yes

Is the study design appropriate and does the work have academic merit?
Yes

Are sufficient details of methods and analysis provided to allow replication by others?
Yes

If applicable, is the statistical analysis and its interpretation appropriate?
Yes

Are all the source data underlying the results available to ensure full reproducibility?
Yes

Are the conclusions drawn adequately supported by the results?
Yes

Competing Interests: No competing interests were disclosed.

Reviewer Expertise: Quantum optics

I confirm that I have read this submission and believe that I have an appropriate level of expertise to confirm that it is of an acceptable scientific standard.

Author Response 22 Sep 2021

Anna Bychek

Dear Dr. Petrosyan, First, we would like to thank you for a thorough reading of the manuscript and your constructive and helpful comments. Below, we respond to your comments point by point (our answer is marked using a different font).

1. *… the authors show that detrimental effects of the cavity dephasing can be significantly reduced by introducing additional atomic dephasing. This is a counterintuitive result and it would be useful if the authors can provide a physical interpretation for it.* It is well established that incoherent pumping is always tied to atomic dephasing.

   It reduces the coherence stored in the atoms but at the same time broadens the individual atomic emission line and the spectral gain profile. In this way the effective coupling between atoms with different detunings is enhanced. As can also be seen from Eq. 8, pumping and atomic dephasing appear at the same place in the diagonal terms of the matrix *A* and induce similar physical effects. Thus, individual atomic dephasing spreads the atomic gain over a larger frequency range and thus can induce extra atom-atom coupling by enlarging the overlap of distinct spectral lines, which finally leads to better synchronization. We added a corresponding sentence in our paper.

2. *On page 6, in the third paragraph, it should be the Gaussian distribution of the atomic transition frequencies σ=κ (not atom number).*

   Here the Gaussian distribution is related to Figure 4(a) (Figure 3(a) in the original version), where we consider only five frequency clusters $\Delta_m=[-\kappa; -\kappa/2; 0; \kappa/2; \kappa]$. On this simple example we would like to introduce our concept to the readers, which we use throughout the whole paper. We model the continuous frequency distribution with the standard deviation σ by choosing equidistant cluster frequencies, with the number of atoms per cluster given by the Gaussian distribution with the standard deviation σ. We have slightly changed the text in order to make it straightforward.

3. *In Fig. 4 perhaps it should be stated that what is shown is the normalized intensity (or max intensity is set to 1).*

   The comment on this has been added to the figure caption.

4. *In Fig. 5, the critical value of the pumping rate vs inhomogeneous linewidth σ is shown for $N=10^4$ atoms. How does it change with the atom number? Is it approximately inversely*

proportional to N, for otherwise the same parameters?

This result appears to be independent of *N*. For an arbitrary given number of *N* the structure of the spectrum remains the same as in Fig. 5 (Fig. 4 in the original version). In a weak pumping regime, the number of atoms determines the linewidth of each individual line, while after merging it determines the linewidth of a single line. The critical value of pumping which distinguishes the discrete spectrum of clusters from a single line remains the same for any *N*. In the revised version, we give a comparison of the results for $N=10^2$ (red dots) and $N=10^4$ (blue circles) atoms in Fig.6, which confirms our predictions.

5. *The authors mention that inhomogeneous broadening of the atomic transition can originate from trapping within the cavity, motion, or optical pumping. It would be useful if the author could expand the discussion a bit and clarify these issues. Do they mean that, depending on the atom position in the trap, its transition frequency will be different? Or that atom motion leads to the Doppler frequency shift? How does optical pumping lead to different frequency shifts of different atoms?*

   Here we mainly think of differential Stark shifts e.g. induced by an inhomogeneous trapping lattice or a pump laser with with a Gaussian profile. Actually several lasers are needed to implement an effective pump rate as we use here. We are currently quantitatively studying this effect in a separate effort, which we hope will be finished soon. Doppler shifts would have similar effects in ring cavities, whereas in a standing-wave cavity they would generate a time-dependent atom-field coupling which we do not consider here.

6. *The present manuscript did not explicitly consider high-density atomic ensembles, where the atomic collisions or dipole-dipole interactions would lead to increased coherence relaxation or collective broadening of the atomic transition. Would the authors be in a position to give some informed speculations as to whether the superradiant lasing with narrow line spectrum will still be possible in such situations, compensating thereby the detrimental broadening due to the interatomic interactions?*

   We are considering atoms at fixed positions here, so that direct collisions are left out. The effect of dipole-dipole interactions have been studied in a small scale, full quantum model before (T Maier *et al.* 2014 Optics express 22 (11), 13269-13279) and appear not too detrimental. Similarly, collisions could have even a positive effect for synchronization, but a quantitative guess is hard (B Zhu *et al*. 2015 New J. Phys. 17 083063). We added those references and a comment.

Competing Interests: No competing interests were disclosed.

Reviewer Report 07 July 2021

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Kamanasish Debnath
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In the paper "*Superradiant lasing in inhomogeneously broadened ensembles with spatially varying coupling*", the authors numerically study superradiant lasing from an incoherently pumped two-level atomic ensemble coupled to a bad cavity. The authors employ second-order mean-field theory and investigate the effect of various noise sources, inhomogeneous broadening, and variable coupling strengths. After a brief introduction and description of the model, the authors describe the second-order cumulant expansion approach which is followed by the results for a single atomic ensemble in the absence of inhomogeneous broadening but with non-zero cavity and atomic dephasing. Finally, the authors introduce inhomogeneous broadening and variable light-matter coupling in their system and study the emission properties.

I think the results are interesting and is of interest to both theoretical and experimental groups working on superradiant lasing using ultra-cold atoms. In addition, a detailed study of various noise sources and inhomogeneous light-matter coupling as considered in this manuscript was not taken into account in Ref. [11] and [31], so the manuscript definitely fills that gap. I have some minor comments which I believe will improve the clarity of the manuscript.

1. Since the manuscript emphasizes on various noise sources, I think it would be helpful to include a color plot for the linewidth of the emission where cavity and atomic dephasing is varied along x and y axis with other parameters remaining the same in Fig.2. This would highlight the tolerance of the linewidth on the two noise sources.

   In addition, the two paragraphs above Fig. 2 where the authors describe the figures was a bit confusing to me, mostly because all the figures were not labelled (a)-(h) and all figures were not described. For example, the authors write- "In Figure 2(c), we scan the linewidth over the coupling strength g ....", which is actually the lower panel of Fig. 2(c). So I would recommend that a proper labelling of all the figures would make it easy for the readers.

2. Next, the authors introduce inhomogeneous broadening and show that the atoms synchronize beyond a critical pump strength and for a weak pumping the spectrum reveals a single broad peak for sufficiently large M. It would be interesting to know the cross correlations between atoms in different clusters (let's say between cluster number 1 and 201) as a function of the pump strength. The correlations should be easily available from the second moment equations. I would expect the cross correlations to grow in the synchronized regime, but what about in the regime where the emission reveals a single broad spectrum. Can the authors comment on this?

3. In Fig.4(c), the authors take into account overall atom-number fluctuations. Did the authors introduce another equation for the atom number like dN/dt which models the loss of atoms

from the trap? It might be helpful for the readers to add a few lines about how this was done numerically.

4. Was the equation $R_c = 0.4\sigma$ derived by fitting the data points? The authors may add a comment on that for clarity. The equation is independent of M, but what about N? Can the authors make an intuitive comment about it?

5. The authors model the inhomogeneous broadening and varying light-matter coupling with M=11 and K=5 clusters. Does this mean it is a 2D grid of 11 x 5 dimensions?

Few typographical errors-

1. ħ in Eq.1 is no longer present in any of the later equations.

2. A minus sign is missing in Eq.4

3. In Eq.8, the correlation terms with superscript “st” is not defined in the text.

4. On page 6, first paragraph last line. I guess the authors meant gain “medium”.

Is the work clearly and accurately presented and does it cite the current literature?
Yes

Is the study design appropriate and does the work have academic merit?
Yes

Are sufficient details of methods and analysis provided to allow replication by others?
Yes

If applicable, is the statistical analysis and its interpretation appropriate?
Not applicable

Are all the source data underlying the results available to ensure full reproducibility?
Yes

Are the conclusions drawn adequately supported by the results?
Yes

Competing Interests: No competing interests were disclosed.

Reviewer Expertise: Quantum optics

I confirm that I have read this submission and believe that I have an appropriate level of expertise to confirm that it is of an acceptable scientific standard, however I have significant reservations, as outlined above.

Author Response 22 Sep 2021

Anna Bychek

Dear Dr. Debnath, First, we would like to thank you for a thorough reading of the manuscript and your constructive comments, which helped us to improve our work. Below, we respond to your comments point by point by using a different font.

1. *Since the manuscript emphasizes on various noise sources, I think it would be helpful to include a color plot for the linewidth of the emission where cavity and atomic dephasing is varied along x and y axis with other parameters remaining the same in Fig 2. This would highlight the tolerance of the linewidth on the two noise sources.*

   Thank you for the valuable suggestion - in the revised version we have included an additional figure (Fig. 3) illustrating the influence of both cavity and atomic dephasing on the linewidth.

2. *In addition, the two paragraphs above Fig. 2 where the authors describe the figures was a bit confusing to me, mostly because all the figures were not labelled (a)-(h) and all figures were not described. For example, the authors write- "In Figure 2(c), we scan the linewidth over the coupling strength g ….", which is actually the lower panel of Fig. 2(c). So I would recommend that a proper labelling of all the figures would make it easy for the readers.*

   We agree that the labels in Fig. 2 could be misleading. In the revised version, all plots in Fig. 2 are labeled (a)-(h). *2.*

3. *Next, the authors introduce inhomogeneous broadening and show that the atoms synchronize beyond a critical pump strength and for a weak pumping the spectrum reveals a single broad peak for sufficiently large M. It would be interesting to know the cross correlations between atoms in different clusters (let's say between cluster number 1 and 201) as a function of the pump strength. The correlations should be easily available from the second moment equations. I would expect the cross correlations to grow in the synchronized regime, but what about in the regime where the emission reveals a single broad spectrum. Can the authors comment on this?*

   This is indeed a question which was interesting to us as well. In the revised version, we decided to include an appendix regarding the cross-correlations between atoms in different clusters. We considered the cross-correlations between the first and the central clusters in Figure 5(a) (Figure 4 in the original version) as a function of the pumping strength. In a weak pumping regime, the correlations are zero and grow with the pumping strength, as shown in Figure 9 in the appendix. The function reaches its maximal value when the ensemble is fully synchronized. However, as pumping continues to grow the correlations decrease due-to growing dephasing imposed by pumping.

4. *In Fig.4(c), the authors take into account overall atom-number fluctuations. Did the authors introduce another equation for the atom number like dN/dt which models the loss of atoms from the trap? It might be helpful for the readers to add a few lines about how this was done numerically.*

The total number of atoms always remains constant and equal to N, so there is no loss of atoms form the trap. The particle number fluctuations simply were added in the cluster's atom numbers as small random deviations from a Gaussian distribution in a way that keeps the total number of atoms constant. The comment on this has been added in the text.

5. *Was the equation R_c=0.4σ derived by fitting the data points? The authors may add a comment on that for clarity. The equation is independent of M, but what about N? Can the authors make an intuitive comment about it?* Yes, the function R_c=0.4σ was found as the best linear fit to the data points.

   The comment on this has been added to the figure caption. This result appears to be independent of *N*. For an arbitrary given number of *N* the structure of the spectrum remains the same as in Fig. 5 (Fig. 4 in the original version). In a weak pumping regime, the number of atoms determines the linewidth of each individual line, while after merging it determines the linewidth of a single line. The critical value of pumping which distinguishes the discrete spectrum of clusters from a single line remains the same for any *N*. In the revised version, we give a comparison of the results for $N=10^2$ (red dots) and $N=10^4$ (blue circles) atoms in Fig.6, which confirms our predictions.

6. *5. The authors model the inhomogeneous broadening and varying light-matter coupling with M=11 and K=5 clusters. Does this mean it is a 2D grid of 11 x 5 dimensions?* This is correct. The overall number of clusters in the model is equal to *MxK* clusters of atoms.

7. *Few typographical errors-*

- *ħ in Eq.1 is no longer present in any of the later equations.*
- *A minus sign is missing in Eq.4*
- *In Eq.8, the correlation terms with superscript "st" is not defined in the text.*
- *On page 6, first paragraph last line. I guess the authors meant gain "medium".*

Thank you for the helpful remarks. We agree and corrected the errors, except Eq. 4 which is correct, for a reference see, for example, Eq.(94) in Jacobs K, Steck DA. A straightforward introduction to continuous quantum measurement. Contemporary Physics 47(5):279-303 (2006).

Competing Interests: No competing interests were disclosed.